

# Physics and the Measurement of Continuous Variables\*

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## Abstract

This paper addresses the doubts voiced by Wigner about the physical relevance of the concept of geometrical points by exploiting some facts known to all but honoured by none: Almost all real numbers are transcendental; the explicit representation of any one will require an infinite amount of physical resources. An instrument devised to measure a continuous real variable will need a continuum of internal states to achieve perfect resolution. Consequently, a laboratory instrument for measuring a continuous variable in a finite time can report only a finite number of values, each of which is constrained to be a rational number. *It does not matter whether the variable is classical or quantum-mechanical.* Now, in von Neumann's measurement theory [13], an operator  $A$  with a continuous spectrum – which has no eigenvectors – cannot be measured, but it can be approximated by operators with discrete spectra which are measurable. The measurable approximant  $F(A)$  is not canonically determined; it has to be chosen by the experimentalist. It is argued that this operator can always be chosen in such a way that Sewell's results [19, 20] on the measurement of a hermitian operator on a finite-dimensional vector space (described in Sec. 3.2) constitute an adequate resolution of the measurement problem in this theory. From this follows our major conclusion, which is that the notion of a geometrical point is as meaningful in nonrelativistic quantum mechanics as it is in classical physics. It is necessary to be sensitive to the fact that *there is a gap between theoretical and experimental physics*, which reveals itself tellingly as an error inherent in the measurement of a continuous variable.

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*Mathematical descriptions are necessarily more refined than the physical operations that they purport to represent.*

G. L. Sewell

## Introduction

At almost every step, theoretical physics makes the assumption that space, time and space-time are continua – that is, they are locally homeomorphic with  $\mathbb{R}^3$ ,  $\mathbb{R}$  and  $\mathbb{R}^4$  respectively. If one sets up a cartesian coordinate system on any of them and chooses a point at random, that point will have, *with probability one*, at least one irrational coordinate. Yet all measurements recorded in the laboratory are expressed in terms of rational numbers. The set of rational numbers  $\mathbb{Q}$  is generally considered discrete, i.e., endowed with the discrete topology (in which one-point sets are open).

It is the view of many eminent mathematicians that “*Bridging the gap between the domains of discreteness and of continuity* is a central, presumably even *the* central problem of the foundations of mathematics” ([5], p.211); yet this gap – which is precisely the gap between experimental and theoretical physics mentioned above – does not seem to have attracted attention in physics itself, despite the dependence of physics on mathematics.

The problem reveals itself as soon as one considers infinitely precise measurements of a continuous variable. In this article we shall analyse the problem, suggest a solution and briefly explore some implications of the solution.

The plan of this work is as follows. In Sec. 1 we shall make precise the notion of a ‘perfect’ classical instrument for measuring lengths (the basic continuous variable in physics) in terms of the resources it would demand, and formulate these requirements as finiteness conditions. In Sec. 2 we shall return to the problem of single measurements in quantum mechanics that was emphasized by Wigner [23, 26], and reconsider his conclusions in the light of von Neumann’s original formulation and the finiteness conditions of Sec. 1. In Sec. 3 we shall sketch the main problem of von Neumann’s theory of measurement in quantum mechanics, paying attention to the measurement of operators with continuous spectra, which is often neglected in the literature. After a brief consideration of the results of Hepp [7], we shall proceed to the results recently obtained by Sewell [19, 20]. In Sec. 4, we shall suggest the division of measurement theory into two: i)  *$\mathbb{R}$ -measurement theory*, in which no finiteness conditions are imposed and the problems arising from continuous spectra are addressed via the weakest possible hypothesis; ii)  *$\mathbb{Q}$ -measurement theory*, in which finiteness conditions appropriate to laboratory

physics are imposed, and a selection criterion, again appropriate to laboratory physics, is proposed to address the problems arising from continuous spectra. It will be seen that Sewell's work provides an adequate resolution of the problem in  $\mathbb{Q}$ -measurement theory. In Sec. 5 we shall remark upon some possible implications.

## 1 Measurement of Length

We begin by stating our basic assumptions (in the form of definitions).

**Definition 1.1** *In the following, the terms point and line will be used in the sense of Euclidean geometry. The Euclidean line will be assumed to be homeomorphic with the set of real numbers  $\mathbb{R}$  with its usual topology [17].*

**Definition 1.2** *A length-measuring device will be said to be perfect (or to have infinite resolution) if its states can be brought into  $(1, 1)$  correspondence with points on the unit interval  $[0, 1]$ .*

We now consider the physical (memory) resources that would be required to set up such a device.

Assume that we have a plane on which a line  $l$  is marked. Let  $O$  and  $B$  be fixed points on  $l$ , and  $A$  any point on the segment  $OB$ . We wish to measure the length  $L(A)$  of the segment  $OA$  with a perfect measuring device, the scale of length being the one that is built into the device.

A length measurement may be visualized as a two-step process, as follows:

$$A \mapsto \xi \mapsto \alpha \tag{1}$$

At the first step, one maps the point  $A$  to a state  $\xi$  of the apparatus (for example, by aligning the point  $A$  with the cross-hairs). At the second step, one carries out a read-write operation, i.e., reads the numerical value  $\alpha$  associated with the state  $\xi$  and writes it down. This numerical value will be the length of the segment  $OA$ :  $L(A) = \alpha$ .

The reason for amalgamating the 'read' and 'write' operations into one step is that we are assuming, following Bohr, that a measurement is not completed until it is recorded. However, the two can be analysed separately. A write operation will require a certain amount of computer memory, paper and pencil, or whatever.

If  $A$  is randomly chosen, then  $L(A)$  will, with probability one, be a transcendental number.<sup>1</sup> In any digital representation, a transcendental number

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<sup>1</sup>Recall that a real number is called transcendental if it is not algebraic, i.e., is not the solution of an algebraic equation. The set of all real algebraic numbers is countable, and

has an infinite number of digits. It will require an infinite amount of physical memory to record it, and an infinitely large laboratory to hold the record.

Let us now turn to the read operation, that is the map  $\xi \mapsto \alpha$  in eq. (1). This condition – that the measuring instrument return a numerical value without calling upon devices external to it – is required to eliminate the possibility of infinite von Neumann chains

$$A \mapsto \xi \mapsto \eta \mapsto \cdots \mapsto \alpha \quad (2)$$

To fulfil this condition the ‘reader’ must be equipped with a read-only memory cell for each state of the device, the cell for the state  $\xi_0$  containing the number  $\alpha_0$  appropriate to it. The number of memory cells required is the power of the continuum. Therefore an infinite amount of physical memory will have to be built into the reader.<sup>2</sup> Again, an infinitely large laboratory will be needed to accommodate it.

We may therefore articulate the following:

**Conclusion 1.3** *If the finiteness of physical resources is considered as a constraint, the only physical quantities that can be measured precisely are those that can assume only a finite number of values, each possible value being a rational number.*

It should be emphasized that this conclusion derives solely from the information content of real numbers.

An alternative statement of conclusion 1.3 would be:

**Conclusion 1.4** *A (digital) measuring instrument can have only a finite number of states  $\xi$ , and the measured value  $\alpha_\xi$  associated with the state  $\xi$  must be a rational number.*

These conclusions hold both for classical and quantum-mechanical systems. The concept of a perfect measurements is meaningful only if one is willing to admit instruments that are infinitely large.

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therefore of Lebesgue measure zero on the real line. It follows that the Lebesgue measure of the set of transcendentals in any interval equals the length of the interval.

<sup>2</sup>Observe that the actual size of each memory cell – as long as it is nonzero – is not relevant to the above argument.

## 2 Measurement of Single Observables in Quantum Mechanics

We now turn to considerations that are specific to quantum mechanics. Our discussion will be based on the mathematically rigorous Hilbert space formulation of quantum mechanics that was advanced by von Neumann in his 1932 monograph [13]. We shall therefore understand the terms ‘measurement’ and ‘the theory of measurement in quantum mechanics’ in the sense of von Neumann. In this section we shall review the measurement of single observables, preparatory to a consideration of the main problem of von Neumann’s theory in Sec. 3.

von Neumann’s development of the statistical interpretation of quantum mechanics is based on the analysis of *two successive measurements* of a self-adjoint operator  $A$  upon the same system, the second measurement immediately following the first.<sup>3</sup> It is based on the assumption that *individual measurements can be performed*, and the first question he addresses concerns the accuracy of these measurements. It suffices to consider the following three cases<sup>4</sup> (see [13], pp. 211-220):

1. The spectrum of  $A$  is discrete and nondegenerate. In this case  $A$  can be measured precisely. If the state before measurement is a superposition of eigenvectors of  $A$ , then the state after measurement is a unique eigenvector of  $A$ , which is determined by the measured value.
2. The spectrum of  $A$  is discrete, but degenerate. In this case,  $A$  can be measured precisely, but the state after measurement “is not uniquely determined by the knowledge of the result of the measurement” ([13], p. 218). However, the second measurement will return the same numerical value as the first.<sup>5</sup>

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<sup>3</sup>It is the result of the second measurement that tells us whether the system behaves in accordance with quantum mechanics, or the 1924 radiation theory of Bohr, Kramers and Slater (see [13], pp. 213-214). This assumes that the result of an individual measurement is error-free, or very nearly so. The nineteen-page paper by Bohr, Kramers and Slater [3] contains only one formula,  $h\nu = E_1 - E_2$ , and the modern reader will probably find it difficult to read. It is therefore worth mentioning that the background to this paper is explained in some detail in the Historical Introduction of the volume [21] edited by van der Waerden, in which this paper is reprinted. Recall that in the Bohr-Kramers-Slater theory energy and momentum are conserved statistically, but not in individual events.

<sup>4</sup>These considerations may easily be extended to include operators with mixed spectra.

<sup>5</sup>If there exists a self-adjoint operator  $B$  which commutes with  $A$  and lifts the degeneracy, then  $A$  and  $B$  may be measured simultaneously. The state after such a simultaneous measurement is again a unique joint eigenvector of  $A$  and  $B$ .

3. The spectrum of  $A$  is continuous. In this case,  $A$  *cannot* be measured precisely.<sup>6</sup> However,  $A$  can be measured approximately, in the following sense. Let  $\lambda_k, k \in \mathbb{Z}$  be points on the real line such that  $\lambda_k < \lambda_{k+1}$ . Then one can determine an interval  $(\lambda_n, \lambda_{n+1})$  in which  $A$  lies. We may, without loss of generality, assume that  $\lambda_{k+1} - \lambda_k = \epsilon$  for all  $k$ . Then  $\epsilon$  is the measurement error. Finally, for each  $k \in \mathbb{Z}$ , let  $\lambda^k$  be any point in  $(\lambda_k, \lambda_{k+1})$ . Then there exists an operator  $F(A)$  with a discrete spectrum that consists precisely of the points  $\lambda^k, k \in \mathbb{Z}$  such that an approximate measurement of  $A$  is equivalent to a precise measurement of  $F(A)$  ([13], pp. 220-221).<sup>7</sup>

Thus von Neumann reduced approximate measurements of operators with continuous spectra to exact measurements of operators with discrete spectra. This reduction was based on the mathematical results that a self-adjoint operator  $A$  with (only) a continuous spectrum had no eigenvectors, but a plentiful supply of approximate eigenvectors ( $\|A\phi - \lambda\phi\| < \epsilon$ ).<sup>8</sup> However, from 1952, Wigner began to express reservations about the very notion of exact measurement in von Neumann's theory. In his 1981 lecture notes he wrote ([22], p. 298):

“Unfortunately, as we shall see, there are serious limitations on the measurability of an arbitrary quantity. They blur the mathematical elegance of von Neumann's original postulate that all self-adjoint operators are measurable<sup>9</sup>... What then are the limitations of measurability?

*Only Quantities Which Commute with ALL Additive Conserved Quantities Are Precisely Measurable.”*

The first result in this direction was obtained by Wigner in 1952 [23]. The conserved quantity was the  $z$ -component of the angular momentum; the quantity measured was the  $x$ -component of the spin of a spin- $\frac{1}{2}$  particle. In 1960, Araki and Yanase proved that a bounded self-adjoint operator

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<sup>6</sup>This is, in some sense, obvious; eigenvectors belonging to the continuous spectrum (such as plane waves) do not lie in the Hilbert space.

<sup>7</sup> We shall call such an  $F(A)$  a *von Neumann approximant* of  $A$ .

<sup>8</sup>This fact forms the basis of the spectral theorem for bounded (and eventually unbounded) self-adjoint operators; see, for example, [15] or [14].

<sup>9</sup>Wigner's use of the word ‘all’ in this sentence is at odds with what he had written earlier in the same notes. On p. 274 of [22], he wrote: “We now proceed to the more general case in which  $A$  may also have a continuous spectrum. In this case one has to admit that measurement will not yield a mathematically precise value – no one asks whether the outcome of this measurement is a rational or irrational number.”

$\mathfrak{M}$  with discrete spectrum which does not commute with an additively conserved quantity cannot be measured precisely [1]. They also showed that an approximate measurement could be carried out, that is, the measurement error could be made smaller than any  $\epsilon > 0$ , provided that the measuring apparatus was a superposition of sufficiently many eigenstates, with different eigenvalues, of the conserved quantity.<sup>10</sup>

Let us now consider, briefly, the measurement of position of a point-particle not bound to a site. For such a particle the position operators  $\vec{X}$ , if they exist, will surely have continuous spectra, and will therefore be only approximately measurable. From von Neumann's argument, one concludes that there will exist three exactly-measurable operators  $\vec{X}_\Delta$  with discrete spectra such that the exact measurement of  $\vec{X}_\Delta$  is equivalent to a determination of the position of the particle within a parallelepiped of volume  $\Delta$ .

In relativistic theories, position operators (such as the Newton-Wigner position operators) may exhibit acausal behaviour. Hegerfeldt has shown that, in a theory in which one-particle states belong to irreducible representations of the inhomogeneous Lorentz group with  $m \geq 0$ , the existence of such an operator contradicts the principle of causality ([6]; references to earlier works may be found there).

Wigner was greatly upset by this result. In [22], p. 312 he wrote:

“No matter how one defines the position, one has to conclude that the velocity, defined as the ratio of two subsequent position measurements divided by the time interval between them, has a finite probability of assuming an arbitrarily large value, exceeding  $c$ . One either has to accept this, or deny the possibility of measuring the position precisely, or even giving significance to this concept; a very difficult choice!”

However, it is also possible to view the facts<sup>11</sup> in a more conservative manner; namely, that on a matter of such import, one should withhold judgment until it has been determined whether or not nonrelativistic quantum mechanics can stand as an autonomous, consistent physical theory. In the latter endeavour, the most important ingredient that has been missing is (or so the author believes) a resolution of the measurement problem.

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<sup>10</sup>In 1961, Yanase returned to the case considered by Wigner in 1952 and obtained a lower bound for the error  $\epsilon$  in terms of the “size” of the apparatus, defined as the mean square of the conserved quantity over the apparatus states [27]. This result was attributed by Wigner to Araki and Yanase in [22], p. 304.

<sup>11</sup>The facts here are the problems of relativistic quantum theories, which are necessarily field theories.

Assume that the measurement problem has been resolved. Let now  $A, F(A)$  and  $\epsilon$  be as earlier, and let  $H$  be an additively conserved quantity that does not commute with  $A$ . Assume further that  $F(A)$  is bounded, and that it does not commute with  $H$ . Then, according to the results of Araki and Yanase,  $F(A)$  can be measured approximately, within an error  $\epsilon$ , for any  $\epsilon > 0$ . An approximate measurement of  $F(A)$  is equally an approximate measurement of  $A$ ; the fact that  $F(A)$  cannot be measured precisely does not seem to change the situation qualitatively. Whether this qualitative picture stands up to quantitative scrutiny can only be discussed in the context of a resolution of the measurement problem. The problem, and its resolution by Sewell, are summarized below.

### 3 The Problem of Measurement in Quantum Mechanics

As is well known, the answer offered by the von Neumann theory of measurement raises a problem of considerable gravity: Schrödinger's cat paradox.

In von Neumann's theory, the measuring device is regarded as an assembly of microscopic quantum systems. There is no characterization of the states of the device other than as vectors (or perhaps subspaces) of a Hilbert space. Begin by considering, as Wigner does in [24] and [26], the case when the state of the system is an eigenstate of the operator being measured.

Let  $\mathfrak{H}$  and  $\mathfrak{K}$  be the Hilbert spaces of the system and the apparatus respectively. Suppose that  $A$  is the operator being measured, the system has been prepared in an eigenstate  $\sigma^{(\nu)} \in \mathfrak{H}$  of  $A$ , and the initial state of the apparatus is  $a \in \mathfrak{K}$ . The initial state of the combined system is then  $a \otimes \sigma^{(\nu)} \in \mathfrak{K} \otimes \mathfrak{H}$ . The interaction will change only the state of the apparatus:

$$a \otimes \sigma^{(\nu)} \rightarrow a^{(\nu)} \otimes \sigma^\nu \quad (3)$$

The above transition, unlike the reduction of the wave packet, may be effected by a unitary time evolution.<sup>12</sup> But all it does is to mirror the state of one quantum-mechanical system by another quantum-mechanical system. How is the state of the latter to be determined? If the apparatus is an object of greater complexity than the system, the problem of determining its state may be even more complex. Coupling a second instrument to determine the state of the first will be of little help if the second instrument is also a quantum-mechanical system like the first.

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<sup>12</sup>This equation is eq. (1) of Wigner's notes ([22], p. 328), and also of his 1963 paper [24].



This example clearly shows that the ‘quantum-mechanical measurement problem’ consists of two distinct problems. The first is the problem of individual measurements: if the system is known to be in an eigenstate of a certain observable, how does the measurement reveal which eigenstate it is in? Without solving this problem, one cannot distinguish, by experiment, between quantum mechanics and the Bohr-Kramers-Slater theory. The theoretical problem of explaining the reduction of the wave packet, that is, of explaining the result of *many* measurements upon identical copies of the system, becomes well-posed only after the problem of individual measurements has been resolved.

von Neumann’s conscious ego hypothesis solves both of these problems. It is, however, what mathematicians would call a very strong hypothesis. One would like to obtain the same results under somewhat weaker conditions.

### 3.1 The infinite-system approach

The quantum theory of systems with infinitely many degrees of freedom was developed in the 1960’s. It was based, not on Hilbert space but on  $C^*$ - and  $W^*$ -algebras (see, for instance, [18]). For  $N$ -particle systems, the canonical commutation (and anticommutation) relations had only one irreducible representation. But for a countable infinity of degrees of freedom, there were many inequivalent irreducible representations, as well as representations that were reducible but not fully reducible (see [16], and references cited therein).<sup>13</sup> Use of operator-algebraic methods established a measure of control over these representations. It was shown by Lanford and Ruelle that, for any  $*$ -representation  $\pi$  of the algebra of observables  $\mathfrak{A}$ , there existed an algebra that could be interpreted as the algebra of observables measurable outside any bounded region of space [11]. They called these *observables at infinity*. Typically, these observables are global (spatial) averages of local ones. (The observables at infinity could all be multiples of the identity; if this was indeed the case, the representation  $\pi$  was said to have *short-range correlations*.)

Hepp realized that individual measurements could be regarded as determining the values of observables at infinity of infinite quantum-mechanical systems. He then attempted to exploit this fact to eliminate the conscious ego from von Neumann’s measurement theory. In 1972, using these observables as ‘pointers’, he constructed a scheme in which the interaction between the measuring device and the observed system caused (i) the pointer value

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<sup>13</sup>Only one of these representations contained a no-particle state, and could be handled by Fock space methods.

to change, and (ii) the state vector of the observed system to collapse [7]. However, these changes were only effected in the limit  $t \rightarrow \infty$ ; they could not be effected in finite times.

The reason is as follows. The pointer positions of the apparatus are necessarily different in the initial and final states of the total system (observed system plus apparatus). This implies that the primary representations<sup>14</sup> associated with the initial and final states are unitarily inequivalent to each other. For *finite times*, time evolution is unitarily implemented in the Hilbert space of the primary representation associated with the initial state of the total system, which means that the pointer remains in its null position. A change of pointer position requires a change of representation. While this could not be achieved in finite time, Hepp succeeded in showing that it could be achieved in the limit  $t \rightarrow \infty$ .

In 1975, Bell claimed that Hepp's results were not valid, because the limit " $t \rightarrow \infty$  never comes" [2]. Although his argument was flawed,<sup>15</sup> the point he raised deserves consideration.

Experimental physics is constrained by the finiteness of laboratory size and available time. To be significant, (theoretical) results that are obtained in the infinite volume and/or infinite time limit must satisfy one further condition, namely the impossibility of distinguishing, *experimentally*, between the limiting state and states at *all* sufficiently large but finite  $(t, V)$ .<sup>16</sup> The requirement that time evolution be a one-parameter group of automorphisms of the algebra supplies the required stability, but does not provide an estimate of the rapidity of convergence to the limit.

### 3.2 The finite-system approach

The finite-system approach developed by Sewell is based on schemes devised by van Kampen in 1954 [8] (see also [9] and [10]) and Emch in 1964 [4] for deriving the Pauli master equation (or a generalization of it). van Kampen was more concerned with the physics of coarse-graining, i.e., with understanding how microscopic observables gave rise to macroscopic ones, and how at the same time irreversibility arose from the underlying time-reversible classical or quantum mechanics. Emch was more concerned with finding a mathematically rigorous framework that underlay the master equation. He wrote explicitly that the "difficult problem...of how to determine in a natural

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<sup>14</sup>Recall that a representation of the algebra  $\mathfrak{A}$  is called *primary* if its centre is trivial, and these include the irreducibles.

<sup>15</sup>It was based on a model for which the Schrödinger picture does not exist. This fact was also overlooked by the editors of [22] (see their remarks on p. 782 of [22]).

<sup>16</sup>In our opinion, this requirement as important as that of mathematical rigour.

way [the macroscopic observables] from an a priori given [set of microscopic observables] will not be touched upon...". Sewell's analysis accepts the rigorous framework of Emch, but his macroscopic observables are related to microscopic ones as envisaged by van Kampen.

Sewell's synthesis of the van Kampen-Emch schemes may be described as follows:

1. The instrument  $\mathcal{I}$  is an object consisting of  $N$  particles, governed by quantum mechanics. A pure quantum-mechanical state of the instrument is a vector in an infinite-dimensional Hilbert space  $\mathfrak{K}$ . Since  $N$  is finite, von Neumann's uniqueness theorem applies.
2. The full algebra  $\mathcal{B}$  of observables of  $\mathcal{I}$  contains an Abelian subalgebra  $\mathcal{M}$ . The elements of  $\mathcal{M}$  are macroscopic observables in the sense of van Kampen.
3. There are no superselection rules on  $\mathfrak{K}$ . (This requirement is stated explicitly in Emch [4].) This means that the centre of  $\mathcal{B}$  consists of multiples of the identity. Put differently, given any macroscopic observable  $M \in \mathcal{M}$ , there is an observable  $B \in \mathcal{B} \setminus \mathcal{M}$  which does not commute with  $M$ .<sup>17</sup>
4. The spectra of the macroscopic observables  $M \in \mathcal{M}$  are discrete. The Hilbert space  $\mathfrak{K}$  decomposes into a set of pairwise-orthogonal subspaces, each of which is the simultaneous eigenspace of every observable in  $\mathcal{M}$ . These subspaces are the quantum analogues of classical phase cells, and their dimensionalities are astronomically large.

Under these conditions, Sewell showed that the essential conclusions of Hepp's analysis are reproducible on the laboratory scale; that is, in a finite time, and with the microscopic system  $\mathcal{S}$  coupled to an apparatus  $\mathcal{I}$  of large but finite size [19, 20].  $\mathcal{I}$  is as described above, the Hilbert space of  $\mathcal{S}$  is  $\mathfrak{H}$  and the Hamiltonians of  $\mathcal{S}$  and  $\mathcal{I}$  are  $H$  and  $K$  respectively. The coupled system is conservative, its Hilbert space is  $\mathfrak{H} \otimes \mathfrak{K}$ ,<sup>18</sup> and its total Hamiltonian  $H_c = H \otimes I_{\mathfrak{K}} + I_{\mathfrak{H}} \otimes K + V$ . Here  $V$  is the interaction between  $\mathcal{S}$  and  $\mathcal{I}$ , and  $I_{\mathfrak{H}}, I_{\mathfrak{K}}$  the identity operators on  $\mathfrak{H}$  and  $\mathfrak{K}$  respectively. Dynamics of the coupled system is governed by the standard  $N$ -particle Schrödinger equation.

Sewell assumes that the Hilbert space  $\mathfrak{H}$  is  $n$ -dimensional.<sup>19</sup> Then the

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<sup>17</sup>Had the macroscopic observables commuted with every microscopic observable, the Hilbert space  $\mathfrak{K}$  would have split into superselection sectors, and no observable, either microscopic or macroscopic, would have been able to induce a transition from one sector to another.

<sup>18</sup>Sewell's ordering of the factors  $\mathfrak{H}$  and  $\mathfrak{K}$  is the opposite of Wigner's eq. (3) in [24].

<sup>19</sup>As will become clear later, this assumption cannot be relaxed.

eigenfunctions  $u_r$  of  $H$ ,  $Hu_r = \epsilon_r u_r$ ,  $r = 1, \dots, n$  form a complete orthonormal set in  $\mathfrak{H}$ . The microscopic observables of  $\mathcal{S}$  are assumed to form an algebra  $\mathcal{A}$  of bounded operators on  $\mathfrak{H}$ .

Furthermore,  $\mathcal{M}$  is assumed to consist of linear combinations of a finite set of orthogonal projectors  $\{\Pi_\alpha | \alpha = 1, 2, \dots, \nu\}$  that span  $\mathfrak{K}$ . Then any element  $M \in \mathcal{M}$  can be written in the form  $M = \sum_{\alpha=1}^{\nu} M_\alpha \Pi_\alpha$ , where the  $M_\alpha$  are constants. The subspaces  $\mathfrak{K}_\alpha = \Pi_\alpha \mathfrak{K}$  of  $\mathfrak{K}$  correspond to classical phase cells. Each such cell represents a macrostate of  $\mathcal{I}$ , and is identified by the position of a pointer (or set of pointers) in a measurement process. We shall call  $\alpha$  the *pointer reading*.

The measuring instrument is so designed that the  $\mathcal{S} - \mathcal{I}$  coupling does not induce transitions between the eigenstates  $u_r$  of  $\mathcal{S}$ . When this holds, the interaction  $V$  will take the form

$$V = \sum_{r=1}^n P(u_r) \otimes V_r, \quad (4)$$

where  $P(u_r)$  is the projection operator for  $u_r$  and the  $V_r$  are observables of  $\mathcal{I}$ . It follows from these that the Hamiltonian of the coupled system has the form  $H_c = \sum_{r=1}^n P(u_r) \otimes K_r$ , with  $K_r = K + V_r + \epsilon_r I_{\mathfrak{K}}$ .

The system  $\mathcal{S}$  and apparatus  $\mathcal{I}$  are prepared separately in the initial states  $\psi = \sum_r c_r u_r$  and  $\Omega$  respectively,  $\psi$  being a pure normalized state ( $\sum_r |c_r|^2 = 1$ ) and  $\Omega$  a density matrix.  $\mathcal{S}$  and  $\mathcal{I}$  are coupled at  $t = 0$ , so that the initial state of the coupled system is  $\Phi(0) = P(\psi) \otimes \Omega$ . Its state at time  $t > 0$  is then given by  $\Phi(t) = U^*(t)\Phi(0)U(t)$ , where  $U(t) = \exp(iH_c t)$ . Owing to the form of  $H_c$ , the state  $\Phi(t)$  can be written as

$$\Phi(t) = \sum_{r,s=1}^n \bar{c}_r c_s P_{r,s} \otimes \Omega_{r,s}(t) \quad (5)$$

where  $P_{r,s}$  is the operator in  $\mathfrak{H}$  defined by  $P_{r,s}f = (u_s, f)u_r \forall f \in \mathfrak{H}$ , and  $\Omega_{r,s}(t) = U_r^*(t)\Omega U_s(t)$ , with  $U_r(t) = \exp(iK_r t)$ .

For  $t > 0$ , the time-dependent expectation value of the observable  $A \otimes M$  of the coupled system is, by definition,  $E(A \otimes M) = \text{Tr}(\Phi(t)[A \otimes M])$ . In particular,  $E(A) = E(A \otimes I_{\mathfrak{K}})$ , and  $w_\alpha = E(I_{\mathfrak{H}} \otimes \Pi_\alpha)$  is the probability that  $\mathcal{I}$  is found in the macrostate  $\mathfrak{K}_\alpha$ . Sewell further showed that, owing to the Abelian character of  $\mathcal{M}$ , the expectation functional  $E$  is compatible with a unique conditional expectation functional on  $\mathcal{A}$  with respect to  $\mathcal{M}$ . This functional has the form

$$E(A|\mathcal{M}) = \sum_{\alpha} \omega_\alpha(A) \Pi_\alpha. \quad (6)$$

With this preparation, Sewell showed [20] that  $E(A)$  and  $E(A|\mathfrak{K}_\alpha)$  could be written as

$$E(A) = \sum_{r=1}^n |c_r|^2 (u_r, Au_r) + \sum_{r \neq s; r, s=1}^n \sum_{\alpha=1}^{\nu} F_{r,s;\alpha} \bar{c}_r c_s (u_r, Au_s) \quad (7)$$

and (for  $w_\alpha \neq 0$ )

$$E(A|\mathfrak{K}_\alpha) = \sum_{r,s=1}^n F_{r,s;\alpha} \bar{c}_r c_s (u_r, Au_s) / w_\alpha, \quad (8)$$

where the coefficients  $F_{r,s;\alpha}$  are defined by

$$F_{r,s;\alpha} = \text{Tr} (\Omega_{r,s}(t) \Pi_\alpha). \quad (9)$$

They satisfy the following conditions: i)  $F_{r,s;\alpha} = \bar{F}_{s,r;\alpha}$ ; ii)  $0 \leq F_{r,r;\alpha} \leq 1$ ; and iii)  $\sum_{\alpha=1}^n F_{r,r;\alpha} = 1$ . It follows from these that, for  $z_1, \dots, z_n \in \mathbb{C}$ , the sesquilinear form  $\sum_{r,s=1}^n \bar{z}_r z_s F_{r,s;\alpha}$  is positive, from which it follows that

$$F_{r,r;\alpha} F_{s,s;\alpha} \geq |F_{r,s;\alpha}|^2. \quad (10)$$

The time evolution of the composite system is carried entirely by the  $F_{r,s;\alpha}$ .

With an instrument  $\mathcal{I}$  designed for the purpose, a pointer reading  $\alpha$  should specify a unique microstate  $u_r$  of  $\mathcal{S}$ , and different microstates of  $\mathcal{S}$  should give different pointer readings; the map  $\Gamma$  from the set of microstates  $r$  of  $\mathcal{S}$  to the set of macrostates  $\alpha$  of  $\mathcal{I}$  should be bijective (which requires  $\nu = n$ ). When this holds, the sum  $\sum_{\alpha} F_{r,r;\alpha}$  reduces to the single term  $F_{r,r;\alpha(r)}$  (where  $\alpha(r) = \Gamma(r)$ ), so that  $F_{r,r;\alpha(r)} = 1$  and  $F_{s,s;\alpha(r)} = 0$  for  $s \neq r$ . It now follows from the positivity condition (10) that

$$F_{r,s;\alpha} = 0 \text{ for } r \neq s. \quad (11)$$

Then (7) reduces to

$$E(A) = \sum_{r=1}^n |c_r|^2 (u_r, Au_r), \quad (12)$$

which shows that the wave packet has collapsed. Finally, setting  $A = I_{\mathfrak{H}}$  in (8) one finds that  $w_{\alpha(r)} = |c_r|^2$ , so that (8) becomes

$$E(A|\mathfrak{K}_{\alpha(r)}) = (u_r, Au_r) \quad (13)$$

which shows that the state of  $\mathcal{S}$  following the measurement is the vector state  $\psi = u_r$ .

The interaction between the system  $\mathcal{S}$  and the instrument  $\mathcal{I}$  must be such that the correspondence between microstates of  $\mathcal{S}$  and the macrostates of  $\mathcal{I}$  – namely the bijective map  $\Gamma$  – is stabilized within a finite interval  $\tau$  which is just a microscopic observational time. This is realized in the finite Coleman-Hepp model studied by Sewell [19].

### 3.3 The effect of $n$ upon the quality of $\mathcal{I}$

The instruments  $\mathcal{I}$  that we have been considering so far, in which eq. (11) is strictly valid, have been called *ideal* by Sewell; they do not admit of measurement errors. An ideal instrument will be a useful analytical tool only if it can be approximated sufficiently well in the laboratory. Sewell investigated this question via a notion of *normal* instruments, which took the possibility of measurement errors into account. When this was done, eq. (11) was replaced by one that had an error term,

$$0 < 1 - F_{r,r;\alpha(r)} < \eta(N). \quad (14)$$

The error  $\eta(N)$  should be a strongly-decreasing function of  $N$  which should tend to zero as  $N \rightarrow \infty$ .

In an exactly-soluble model that Sewell considered (the finite Coleman-Hepp model), he found that  $\eta(N) = \exp(-cN/n)$ , where  $c$  is a positive constant of order unity and  $n = \dim \mathfrak{H}$ . As  $n$  was small and fixed ( $n = 2$  in the Coleman-Hepp model) and  $N \sim 10^{24}$ , the error term was utterly negligible. However, if  $n$  increased while  $N$  was held fixed, the arguments that led to the formula  $\eta(N) = \exp(-cN/n)$  could no longer be carried through. To get an idea of how the estimate for  $\eta$  might be affected, consider a measurement in which the pointer reading represents the value of an intensive variable  $v$  of  $\mathcal{I}$ . In that case, each subspace  $\mathfrak{K}_\alpha$  will carry a range  $\sim n^{-1}$  of values of  $v$ , and under the assumptions of the large deviation principle for the fluctuations of macroscopic observables, a pointer reading corresponding to a typical  $\mathfrak{K}_\alpha$  would carry a probability of the order of  $\exp(-cN/n^2)$  that the state of  $\mathcal{I}$  did not lie in that subspace. Thus  $\mathcal{I}$  would not be a reliable instrument when  $n$  comes close to the order of  $N^{\frac{1}{2}}$ .<sup>20</sup>

### 3.4 Reconsideration of the measurement of continuous variables

As pointed out by von Neumann, an observable  $A$  that has only a continuous spectrum cannot be measured precisely, because it has no eigenvalues. Write

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<sup>20</sup>I am indebted to the referee for pointing out the above.

its spectral decomposition as

$$A = \int_{-\infty}^{\infty} \lambda dE_{\lambda}.$$

Then any vector in any subspace  $\mathfrak{H}_{\Delta\lambda} = (E_{\lambda+\Delta\lambda} - E_{\lambda}) \cdot \mathfrak{H}$  of the system Hilbert space  $\mathfrak{H}$  is an approximate eigenvector of  $A$  near the spectral value  $\lambda$ . This fact makes it possible to define an infinity of operators with discrete spectra, each of which can claim to be a measurable approximant to the operator  $A$ . Some further specification is needed to make the notion of *measurement of an operator with a continuous spectrum* well-defined.

The least restrictive specification is clearly the following:

**Hypothesis 3.1** *Let  $A$  be an operator with a continuous spectrum. Then every von Neumann approximant  $F(A)$  to  $A$  should be precisely measurable.*

Accepting this hypothesis means, in Sewell's scheme, admitting every possible value of  $n$ . In this case, as we noted above, Sewell's results would not be applicable.

The alternative to accepting hypothesis 3.1 would be to accept only a suitable subset of the von Neumann approximants  $F(A)$ . The problem is that *there is no canonical choice imposed by the theory*. It would require an external agency to exercise the choice, which would make it difficult to claim that quantum-mechanical measurement theory emerges from quantum mechanics alone. This point is worth repeating, and we present it as a conclusion:

**Conclusion 3.2** *In the formulation of quantum mechanics on Hilbert space, a self-adjoint operator  $A$  with continuous spectrum does not have exact eigenvalues, but only approximate ones. An approximate measurement of  $A$  is defined by von Neumann to be an exact measurement of  $F(A)$ , an operator with discrete spectrum that approximates  $A$ . However, the approximant  $F(A)$  is not uniquely defined by the theory, and has to be chosen by an external agency.*

Our further discussion will be based on the following assumption, which is considerably weaker than von Neumann's conscious ego hypothesis:

**Assumption 3.3** *If the self-adjoint operator  $A$  with a continuous spectrum is at all measurable, its approximant  $F(A)$  which is measured is determined by the design of the experiment.*

This assumption takes cognizance of the following facts:

1. With few exceptions, it is virtually impossible to design, let alone build, an instrument to measure a self-adjoint operator.
2. A laboratory measurement of an unbounded operator such as the momentum can only deal with a bounded subset of its spectrum.
3. *Increasing the resolution of the instrument is usually accompanied by a reduction in the range of the variable measured.* An instrument that is designed to determine the sixth or seventh significant figure after the decimal point will almost surely be designed under the assumption that the first three to five significant figures are known.

In short, *the experimentalist has considerable control on the dimension  $n$  of the Hilbert space  $\mathfrak{H}$  in Sewell's scheme.* It is to be expected that this control will be exercised to ensure the quality of  $\mathcal{I}$ , i.e., to keep  $\exp(-cN/n^2)$  negligibly small. However, even a perfect measurement of  $F(A)$  will only mean that the error in the determination of  $A$  is within the bounds imposed by the definition of  $F(A)$ .

## 4 Two Versions of Measurement Theory

In view of the above, it is our opinion that the subject called measurement theory be divided into two in order to understand the problem and to appreciate the significance of Sewell's results. We shall call these two  $\mathbb{R}$ -*measurement theory* and  $\mathbb{Q}$ -*measurement theory* respectively.<sup>21</sup>

### 4.1 $\mathbb{R}$ -measurement theory

This will be the idealized version in which all finiteness conditions – on physical resources and on the time available to make a measurement – are ignored, and hypothesis 3.1 is accepted. This is the framework that underlies the von Neumann-Wigner theory of measurement in quantum mechanics. Owing to the hypothesis 3.1, Sewell's results do not provide a resolution of the measurement problem in this framework.

$\mathbb{R}$ -measurement theory can only be considered as an analytical tool. One may perhaps be excused for thinking that, as an analytical tool, it has yet to prove its usefulness.

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<sup>21</sup>In mathematics, the symbols  $\mathbb{R}$  and  $\mathbb{Q}$  are used to denote the sets of real and rational numbers respectively.



## 4.2 $\mathbb{Q}$ -measurement theory

In this version of measurement theory, the finiteness of physical resources (and of time available to the experimenter) is taken into account, without sacrificing mathematical rigour. Quantum mechanics is developed on Hilbert space, but it is accepted that an observable that can be observed in the laboratory can assume only a finite number of values, and these values have to be rational. For continuous spectra, this means explicit acceptance of assumption 3.3. This version corresponds more closely to experiment, but departs dramatically from theory; variables that are continuous in the theory are measured as discrete in the laboratory, with the inherent error that this implies. Put differently, the gap between theoretical and experimental physics referred to earlier reappears as the inherent error of measurement. In the opinion of the present author, this should be considered to be a reassuring, rather than a disturbing fact.

In this version of measurement theory, Sewell's results provide an adequate resolution of the measurement problem in quantum mechanics.

## 5 Remarks

1. The 'position operators' that can be measured in the laboratory<sup>22</sup> are defined on finite-dimensional vector spaces, and have the form  $q_M = \sum q_j E_j$ , where the sum is from  $j = 1$  to  $j = J$ , the  $q_j$  are the values that can be returned by the device, and  $E_j$  are one-dimensional projection operators. Clearly,  $q'_M = \sum q'_i E'_i$  is another  $l$ -measurable position operator, and in general one will have  $[q_M, q'_M] \neq 0$ . On the other hand, one may be able to define an  $l$ -measurable momentum operator  $p_M$  that commutes with  $q_M$ ; it is well known that the canonical commutation relations cannot be realized on finite-dimensional vector spaces.
2. Let  $\{E_k | k = 1, \dots, K\}$  be a set of one-dimensional projection operators on a  $K$ -dimensional vector space  $V$  over the complex numbers. The algebra of these projection operators over the real numbers will contain a set of  $l$ -measurable operators  $p$  and  $q$ , and therefore all  $l$ -measurable observables that can be constructed from them. This algebra will clearly be abelian. However, the superposition principle would not have been lost.

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<sup>22</sup>To avoid ambiguity, we shall call such operators *l-measurable*.

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